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UNIVERSITY OF SASKATCHEWAN

MATHEMATICS 124.3 — Final Examination — Solutions

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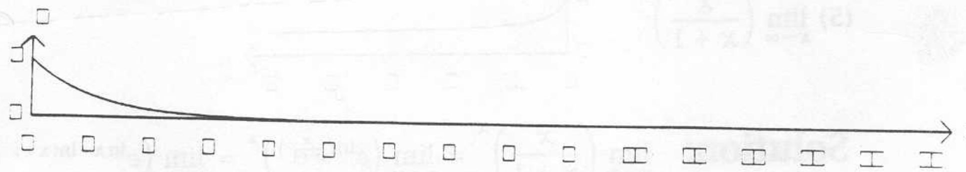
NO BOOKS, NOTES OR CALCULATORS ALLOWED.

The first 28 questions are each worth 2 marks. The last six are worth 3 marks each.

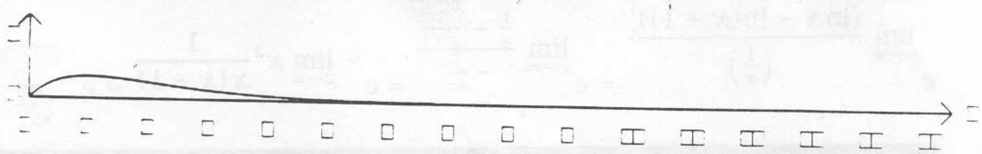
Evaluate the limits:

(1) $\lim_{x\to\infty} e^{-x}$



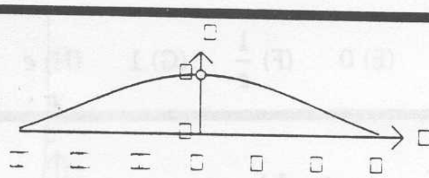
Solution: $\lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \frac{1}{e^x} =$ 0(E)

(2) $\lim_{x\to\infty} xe^{-x}$



Solution: $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} =$ (by L'Hôpital) $\frac{(x)'}{(e^x)'} = \frac{1}{e^x} =$ 0(E)

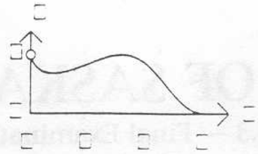
(3) $\lim_{x\to 0} \frac{\sin x}{x}$



Solution: $\lim_{x\to 0} \frac{\sin x}{x} =$ (by L'Hôpital) $\lim_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 =$

1(G)

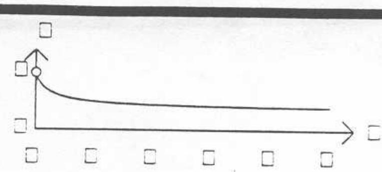
(4) $\lim\limits_{x\to 0^+} x^{\sin x}$



Solution: $\lim\limits_{x\to 0^+} x^{\sin x} = \lim\limits_{x\to 0^+}\left(e^{\ln x}\right)^{\sin x} = e^{\lim\limits_{x\to 0^+} \ln x \sin x} = e^{\lim\limits_{x\to 0^+} \frac{\ln x}{\csc x}}$ =(by L'Hôpital)

$e^{\lim\limits_{x\to 0^+} \frac{(\ln x)'}{(\csc x)'}} = e^{\lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x}} = e^{\lim\limits_{x\to 0^+} \frac{\sin^2 x}{x\cos x}} = e^0 = 1$(G)

(5) $\lim\limits_{x\to\infty}\left(\frac{x}{x+1}\right)^x$



Solution: $\lim\limits_{x\to\infty}\left(\frac{x}{x+1}\right)^x = \lim\limits_{x\to\infty}\left(e^{\ln\left(\frac{x}{x+1}\right)}\right)^x = \lim\limits_{x\to\infty}\left(e^{\ln x - \ln(x+1)}\right)^x =$

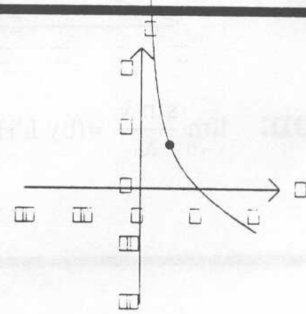
$e^{\lim\limits_{x\to\infty} x[\ln x - \ln(x+1)]} = e^{\lim\limits_{x\to\infty}\frac{\ln x - \ln(x+1)}{\frac{1}{x}}}$ =(by L'Hôpital)

$e^{\lim\limits_{x\to\infty}\frac{(\ln x - \ln(x+1))'}{\left(\frac{1}{x}\right)'}} = e^{\lim\limits_{x\to\infty}\frac{\frac{1}{x}-\frac{1}{x+1}}{-\frac{1}{x^2}}} = e^{-\lim\limits_{x\to\infty} x^2\frac{1}{x(x+1)}} = e^{-1} = \frac{1}{e}$(F)

The possible answers are:

(A) $-\infty$ (B) $-e$ (C) -1 (D) $-\frac{1}{e}$ (E) 0 (F) $\frac{1}{e}$ (G) 1 (H) e (I) ∞ (J)

Find the derivative $f'\left(\frac{1}{2}\right)$ if $f(x) =$ (6) $\sinh\left(\ln\frac{1}{x}\right)$

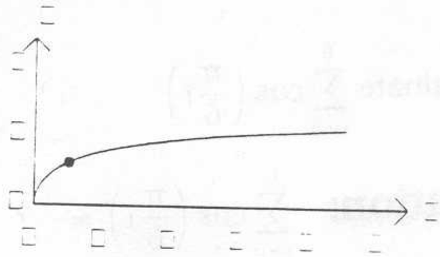


Solution: $f(x) = \sinh(-\ln x) = \frac{e^{-\ln x} - e^{-(-\ln x)}}{2} = \frac{1}{2}\left(\frac{1}{x} - x\right)$, so $f'(x) = \frac{1}{2}\left(-\frac{1}{x^2} - 1\right)$,

and $f'\left(\frac{1}{2}\right) = \frac{1}{2}\left(-\frac{1}{\left(\frac{1}{2}\right)^2} - 1\right) = \frac{1}{2}(-4-1) = -\frac{5}{2}$(B)

2

(7) Find the derivative $f'\left(\frac{1}{2}\right)$ if $f(x) = \tan^{-1}(\sqrt{x})$



Solution: $f'(x) = \frac{1}{1+(\sqrt{x})^2}(\sqrt{x})' = \left(\frac{1}{1+|x|}\right)\left(\frac{1}{2\sqrt{x}}\right)$, so $f'\left(\frac{1}{2}\right) = \left(\frac{1}{1+|\frac{1}{2}|}\right)\left(\frac{1}{2\sqrt{\frac{1}{2}}}\right) =$

$\left(\frac{1}{\frac{3}{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{3}\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{3}$ (G)

The possible answers are:

(A) -5 (B) $-\frac{5}{2}$ (C) $-\sqrt{\frac{5}{2}}$ (D) $-\frac{\sqrt{2}}{3}$ (E) $\frac{1}{3}$ (F) 0 (G) $\frac{\sqrt{2}}{3}$ (H) $\sqrt{\frac{5}{2}}$ (I) $\frac{5}{2}$ (J) 5

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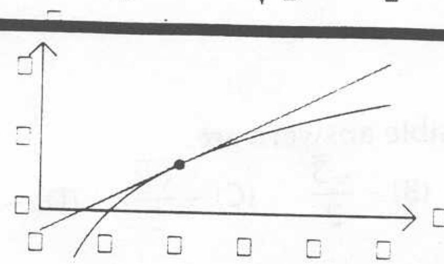
(8) Use differentials to estimate the value of $\ln 2.01$.



Solution: Letting $f(x) = \ln x$, we have $f(2+0.01) \doteq f(2) + f'(2)(0.01)$.

Since $f'(x) = \frac{1}{x}$, we have $f'(2) = \frac{1}{2}$, so $f(2.001) \doteq f(2) + \frac{1}{2}(0.01) = \ln 2 + \frac{1}{200}$

$\ln 2 + \frac{1}{200}$ (D) The possible answers are:

(A) $\ln 2 + \frac{1}{2000}$ (B) $\ln 2 + \frac{1}{1000}$ (C) $\ln 2 + \frac{1}{500}$ (D) $\ln 2 + \frac{1}{200}$ (E) $\ln 2 + \frac{1}{100}$ (F) $\ln 2 + \frac{1}{50}$ (G) $\ln 2 + \frac{1}{20}$ (H) $\ln 2 + \frac{1}{10}$ (I) $\ln 2 + \frac{1}{5}$ (J) $\ln 2 + \frac{1}{2}$

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(9) Evaluate $\sum_{i=0}^{6} \cos\left(\frac{\pi}{6} i\right)$

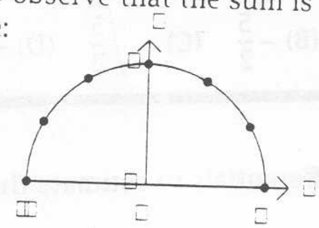
Solution: $\sum_{i=0}^{6} \cos\left(\frac{\pi}{6} i\right) =$

$$\cos\left(\frac{\pi}{6}0\right) + \cos\left(\frac{\pi}{6}1\right) + \cos\left(\frac{\pi}{6}2\right) + \cos\left(\frac{\pi}{6}3\right) + \cos\left(\frac{\pi}{6}4\right) + \cos\left(\frac{\pi}{6}5\right) + \cos\left(\frac{\pi}{6}6\right) =$$

$$\cos 0 + \cos\frac{\pi}{6} + \cos\frac{\pi}{3} + \cos\frac{\pi}{2} + \cos\frac{2\pi}{3} + \cos\frac{5\pi}{6} + \cos\pi =$$

$$1 + \frac{\sqrt{3}}{2} + \frac{1}{2} + 0 + \left(-\frac{1}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right) + (-1) = \quad 0(\mathrm{E})$$

A simple geometric way of getting the same result is to observe that the sum is that of the x-coordinates of the seven points on the unit circle:



The possible answers are:

(A) -1 (B) $-\frac{\sqrt{3}}{2}$ (C) $-\frac{\sqrt{2}}{2}$ (D) $-\frac{1}{2}$ (E) 0 (F) $\frac{1}{2}$ (G) $\frac{\sqrt{2}}{2}$ (H) $\frac{\sqrt{3}}{2}$ (I) 1

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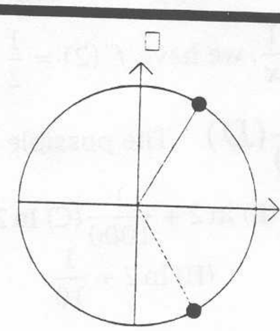
(10) Evaluate $\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^{101}$



Solution: The absolute value of $\frac{1}{2} + \frac{\sqrt{3}}{2} i$ is $\left|\frac{1}{2} + \frac{\sqrt{3}}{2} i\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} =$

$$\sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$$

and the argument is $\theta = \arctan\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \arctan\sqrt{3} = \frac{\pi}{3}$, so

$\frac{1}{2} + \frac{\sqrt{3}}{2} i = 1\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)$ and therefore

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^{101} = 1^{101}\left(\cos\frac{101\pi}{3} + i \sin\frac{101\pi}{3}\right) =$$

$$\cos\left(32\pi+\frac{5\pi}{3}\right)+i\sin\left(32\pi+\frac{5\pi}{3}\right)=$$

$$\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)=\cos\left(-4\frac{\pi}{12}\right)+i\sin\left(-4\frac{\pi}{12}\right):(D)$$

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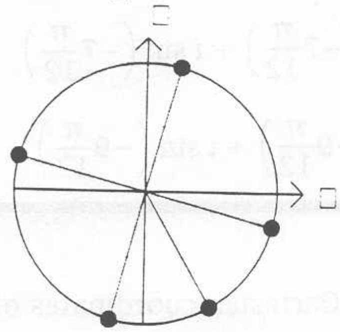
(11) Of the four fourth roots of the complex number $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$, the one lying in the fourth quadrant is:



Solution: The absolute value of $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ is $\left|-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right|=\sqrt{\left(-\frac{1}{2}\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2}=$ $\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{1}=1$

and the argument is $\theta=\arctan\frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}=\arctan\sqrt{3}=\frac{\pi}{3}-\pi=-\frac{2\pi}{3}$, so

$-\frac{1}{2}-\frac{\sqrt{3}}{2}i=1\left(\cos\frac{-2\pi}{3}+i\sin\frac{-2\pi}{3}\right)$ and therefore the fourth roots of $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ are:

$1^{\frac{1}{4}}\left[\cos\left(\frac{\frac{-2\pi}{3}+2\pi k}{4}\right)+i\sin\left(\frac{\frac{-2\pi}{3}+2\pi k}{4}\right)\right]$, $k=0,1,2,3$ or

$\cos\left(\frac{-\pi}{6}+k\frac{\pi}{2}\right)+i\sin\left(\frac{-\pi}{6}+k\frac{\pi}{2}\right)$, $k=0,1,2,3$ or

$\cos\left(\frac{-2\pi}{12}+k\frac{\pi}{2}\right)+i\sin\left(\frac{-2\pi}{12}+k\frac{\pi}{2}\right)$, and taking $k=0$ we get the root in the fourth quadrant, (B)

The possible answers are:

(A) $\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)$ (B) $\cos\left(-2\frac{\pi}{12}\right) + i\sin\left(-2\frac{\pi}{12}\right)$

(C) $\cos\left(-3\frac{\pi}{12}\right) + i\sin\left(-3\frac{\pi}{12}\right)$ (D) $\cos\left(-4\frac{\pi}{12}\right) + i\sin\left(-4\frac{\pi}{12}\right)$

(E) $\cos\left(-5\frac{\pi}{12}\right) + i\sin\left(-5\frac{\pi}{12}\right)$ (F) $\cos\left(-6\frac{\pi}{12}\right) + i\sin\left(-6\frac{\pi}{12}\right)$

(G) $\cos\left(-7\frac{\pi}{12}\right) + i\sin\left(-7\frac{\pi}{12}\right)$ (H) $\cos\left(-8\frac{\pi}{12}\right) + i\sin\left(-8\frac{\pi}{12}\right)$

(I) $\cos\left(-9\frac{\pi}{12}\right) + i\sin\left(-9\frac{\pi}{12}\right)$ (J) $\cos\left(-10\frac{\pi}{12}\right) + i\sin\left(-10\frac{\pi}{12}\right)$

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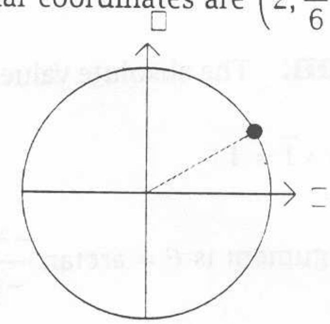
(12) The Cartesian coordinates of the point whose polar coordinates are $\left(2, \frac{\pi}{6}\right)$ are



Solution: With $r = 2$ and $\theta = \frac{\pi}{6}$, we have $x = r\cos\theta = 2\cos\frac{\pi}{6} = 2\frac{\sqrt{3}}{2} = \sqrt{3}$, $y = r\sin\theta = 2\sin\frac{\pi}{6} = 2\frac{1}{2} = 1$, so the correct answer is (F)

The possible answers are:

(A) $(1,1)$ (B) $(1,2)$ (C) $(2,1)$ (D) $\left(\sqrt{2},1\right)$ (E) $\left(1,\sqrt{2}\right)$ (F) $\left(\sqrt{3},1\right)$ (G) $\left(1,\sqrt{3}\right)$

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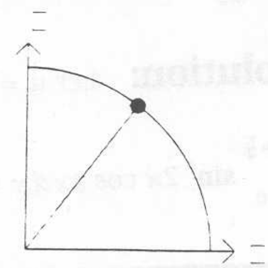
(13) The polar coordinates of the point whose Cartesian coordinates are $(3, 4)$ are



Solution: We have $r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$, and $\theta = \arctan\frac{4}{3}$, so the correct answer is **(H)**

The possible answers are:

(A) $\left(\sqrt{5}, \arctan\frac{3}{5}\right)$ (B) $\left(\sqrt{5}, \arctan\frac{4}{5}\right)$ (C) $\left(\sqrt{5}, \arctan\frac{3}{4}\right)$ (D) $\left(\sqrt{5}, \arctan\frac{4}{3}\right)$

(E) $\left(5, \arctan\frac{3}{5}\right)$ (F) $\left(5, \arctan\frac{4}{5}\right)$ (G) $\left(5, \arctan\frac{3}{4}\right)$ (H) $\left(5, \arctan\frac{4}{3}\right)$

Evaluate the following definite integrals; The possible answers are:

(A) $-\frac{234}{21}$ (B) $-\frac{1}{4}\ln 3$ (C) $-\ln\frac{4}{3}$ (D) $-\frac{1}{16}$ (E) 0 (F) $\frac{1}{16}$ (G) $\ln\frac{2}{3}$ (H) $\ln\frac{4}{3}$ (I) $\frac{1}{4}\ln 3$ (J) $\frac{234}{21}$

(14) $\int_0^3 \sqrt{4 + 7x}\,dx$ **Solution:** Let $u = 4 + 7x$, so that $du = 7dx$, and $dx = \frac{1}{7}du$. Then we have $\int_{x=0}^{x=3} \sqrt{4 + 7x}\,dx = \int_{u=4}^{u=25} u^{\frac{1}{2}}\frac{1}{7}du = \frac{1}{7}\left.\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right|_{u=4}^{u=25} = \frac{2}{21}\left[25^{\frac{3}{2}} - 4^{\frac{3}{2}}\right] = \frac{2}{21}\left[5^3 - 2^3\right] = \frac{2}{21}[125 - 8] = \frac{2}{21}117 = \frac{234}{21}$ (J)

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(15) $\int_0^{\frac{\pi}{4}} \sin^7 2x \cos 2x dx$

Solution: Let $u = \sin 2x$, so that $du = 2\cos 2x dx$, and $dx = \dfrac{du}{2\cos 2x}$. Then

$$\int_{x=0}^{x=\frac{\pi}{4}} \sin^7 2x \cos 2x dx = \int_{u=0}^{u=1} u^7 \cos 2x \frac{du}{2\cos 2x} = \frac{1}{2}\int_{u=0}^{u=1} u^7 du = \frac{1}{2}\frac{u^8}{8}\Big|_0^1 = \frac{1}{16}\text{(F)}$$

(16) $\int_{\ln 2}^{\ln 3} \dfrac{e^{2x} - e^x}{e^{2x} - 1} dx$

Solution: $\int_{\ln 2}^{\ln 3} \dfrac{e^{2x} - e^x}{e^{2x} - 1} dx = \int_{\ln 2}^{\ln 3} \dfrac{e^x (e^x - 1)}{(e^x + 1)(e^x - 1)} dx = \int_{\ln 2}^{\ln 3} \dfrac{e^x}{(e^x + 1)} dx = \ln|e^x + 1|\Big|_{\ln 2}^{\ln 3} =$

$$\ln\left|e^{\ln 3} + 1\right| - \ln\left|e^{\ln 2} + 1\right| = \ln|3+1| - \ln|2+1| = \ln 4 - \ln 3 = \ln\frac{4}{3}\text{(H)}$$

(17) $\int_0^1 \dfrac{1}{x^2 - 4} dx$

Solution: We write $\dfrac{1}{x^2 - 4} = \dfrac{A}{x-2} + \dfrac{B}{x+2}$ and simplify to

$1 = A(x+2) + B(x-2)$. Substituting $x = 2$, we get $A = \frac{1}{4}$, and substituting $x = -2$, we get $B = -\frac{1}{4}$, so we have:

$$\int_0^1 \frac{1}{x^2 - 4} dx = \int_0^1 \frac{1}{4}\left(\frac{1}{x-2} - \frac{1}{x+2}\right) dx = \frac{1}{4}\left(\ln|x-2| - \ln|x+2|\right)\Big|_0^1 = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|\Big|_0^1 =$$

$$\frac{1}{4}\left(\ln\left|\frac{1-2}{1+2}\right| - \ln\left|\frac{0-2}{0+2}\right|\right) = \frac{1}{4}\left(\ln\left|\frac{-1}{3}\right| - \ln|-1|\right) = \frac{1}{4}\left(\ln\frac{1}{3} - 0\right) = -\frac{1}{4}\ln 3\text{(B)}$$

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Evaluate the following definite integrals; The possible answers are:

(A) $\frac{\pi}{6} - \frac{\sqrt{3}}{16}$ (B) $\frac{\pi}{6} + \frac{\sqrt{3}}{16}$ (C) $\ln 2 - \frac{\pi}{12}$ (D) $\ln 2 + \frac{\pi}{12}$

(E) $e^2 - e$ (F) $e^2 + e$ (G) $\frac{e^2-1}{4}$ (H) $\frac{e^2+1}{4}$

(18) $\int_1^e x \ln x dx$

Solution: We use integration by parts:
let $u = \ln x$ and $dv = xdx$ so that $du = \frac{dx}{x}$ and $v = \frac{x^2}{2}$. Then we have

$$\int_{x=1}^{x=e} x \ln x dx = \int_{x=1}^{x=e} udv = uv\Big|_{x=1}^{x=e} - \int_{x=1}^{x=e} vdu = \ln x\frac{x^2}{2}\Big|_{x=1}^{x=e} - \int_{x=1}^{x=e} \frac{x^2}{2}\frac{dx}{x} =$$

$$\ln x\frac{x^2}{2}\Big|_{x=1}^{x=e} - \frac{1}{2}\int_{x=1}^{x=e} xdx = \ln x\frac{x^2}{2}\Big|_{x=1}^{x=e} - \frac{1}{2}\frac{x^2}{2}\Big|_{x=1}^{x=e} = \left(\ln x - \frac{1}{2}\right)\frac{x^2}{2}\Big|_{x=1}^{x=e} =$$

$$\left(\ln e - \frac{1}{2}\right)\frac{e^2}{2} - \left(\ln 1 - \frac{1}{2}\right)\frac{1^2}{2} = \left(1 - \frac{1}{2}\right)\frac{e^2}{2} - \left(0 - \frac{1}{2}\right)\frac{1}{2} = \left(\frac{1}{2}\right)\frac{e^2}{2} + \frac{1}{4} = \frac{e^2+1}{4}(H)$$

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(19) $\int_{\frac{1}{2}}^1 \frac{e^{\frac{1}{x}}}{x^2} dx$

Solution: Let $u = \frac{1}{x}$, so that $du = -\frac{dx}{x^2}$, and $dx = -x^2du$. Then

$$\int_{x=\frac{1}{2}}^{x=1} \frac{e^{\frac{1}{x}}}{x^2}dx = \int_{u=2}^{u=1} \frac{e^u}{x^2}(-x^2du) = -\int_{u=2}^{u=1} e^u du = \int_{u=1}^{u=2} e^u du = e^u\Big|_{u=1}^{u=2} = e^2 - e(E)$$

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(20) $\int_0^{\frac{\pi}{3}} \cos^2 2xdx$ **Solution:** $\int_0^{\frac{\pi}{3}} \cos^2 2xdx = \int_0^{\frac{\pi}{3}} \frac{1+\cos 4x}{2}dx = \frac{1}{2}\left[x + \frac{1}{4}\sin 4x\right]\Big|_0^{\frac{\pi}{3}} =$

$$\frac{1}{2}\left[\frac{\pi}{3} + \frac{1}{4}\sin 4\frac{\pi}{3}\right] = \frac{\pi}{6} + \frac{1}{8}\left(-\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{16}(A)$$

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(21) $\int_{-2}^{1} \frac{2x+5}{x^2+4x+13}dx$

Solution: First complete squares:

$$\int_{-2}^{1} \frac{2x+5}{x^2+4x+13}dx = \int_{-2}^{1} \frac{2x+5}{(x+2)^2+3^2}dx = \int_{-2}^{1} \frac{2(x+2)+1}{(x+2)^2+3^2}dx =$$

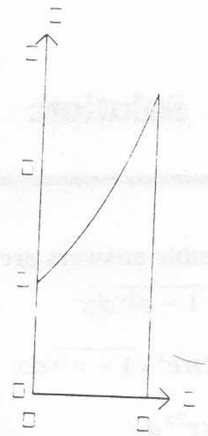
$$\int_{-2}^{1} \frac{2(x+2)}{(x+2)^2+3^2}dx + \int_{-2}^{1} \frac{dx}{(x+2)^2+3^2} \left[\ln\left|(x+2)^2+3^2\right| + \frac{1}{3}\arctan\frac{x+2}{3}\right]\Big|_{-2}^{1} =$$

$$\left[\ln\left|(1+2)^2+3^2\right| + \frac{1}{3}\arctan\frac{1+2}{3}\right] - \left[\ln\left|(-2+2)^2+3^2\right| + \frac{1}{3}\arctan\frac{-2+2}{3}\right] =$$

$$\left[\ln 18 + \frac{1}{3}\arctan 1\right] - \left[\ln 9 + \frac{1}{3}\arctan 0\right] = \ln 18 - \ln 9 + \frac{1}{3}\frac{\pi}{4} - \frac{1}{3}0 = \ln\frac{18}{9} + \frac{\pi}{12} =$$

$$\ln 2 + \frac{\pi}{12}\text{(D)}$$

Let:
$\mathcal{L}$ be the curve $y = f(x) = e^x, 0 \le x \le 1$,
$\mathcal{R}$ be the region lying between $\mathcal{L}$ and the x-axis,
S_x be the surface obtained by rotating $\mathcal{L}$ about the x-axis,
S_y be the surface obtained by rotating $\mathcal{L}$ about the y-axis,
$\mathcal{V}_x$ be the solid obtained by rotating $\mathcal{R}$ about the x-axis,
$\mathcal{V}_y$ be the solid obtained by rotating $\mathcal{R}$ about the y-axis,
$\mathcal{M}_x$ be the moment of $\mathcal{R}$ about the x-axis, and let



Find:

(22) the length of $\mathcal{L}$ **Solution:** (I) $\int_0^1 \sqrt{1 + e^{2x}}\,dx$

(23) the area of $\mathcal{R}$ **Solution:** (J) $\int_0^1 e^x\,dx$

(24) the area of S_x **Solution:** (D) $\int_0^1 2\pi e^x \sqrt{1 + e^{2x}}\,dx$

(25) the area of S_y **Solution:** (C) $\int_0^1 2\pi x \sqrt{1 + e^{2x}}\,dx$

(26) the volume of $\mathcal{V}_x$

Solution: (E) $\int_0^1 \pi e^{2x}\,dx$

(27) the volume of $\mathcal{V}_y$ **Solution:** (F) $\int_0^1 2\pi x e^x\,dx$

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(28) M_x **Solution:** (H) $\int_0^1 \frac{1}{2} e^{2x}\, dx$

The possible answers are:

(A) $\int_0^1 \sqrt{1-e^{2x}}\,dx$ (B) $\int_0^1 \frac{1}{4}e^{2x}dx$ (C) $\int_0^1 2\pi x\sqrt{1+e^{2x}}\,dx$

(D) $\int_0^1 2\pi e^x\sqrt{1+e^{2x}}\,dx$ (E) $\int_0^1 \pi e^{2x}dx$ (F) $\int_0^1 2\pi x e^x dx$

(G) $\int_0^1 xe^{2x}dx$ (H) $\int_0^1 \frac{1}{2}e^{2x}dx$ (I) $\int_0^1 \sqrt{1+e^{2x}}\,dx$

(J) $\int_0^1 e^x dx$

Evaluate the following improper integrals:

(29) $\int_3^7 \frac{dx}{\sqrt{x-3}}$

Solution: Let $u = x-3$. Then we have $\int_{x=3}^{x=7} \frac{dx}{\sqrt{x-3}} = \int_{u=0}^{u=4} u^{-\frac{1}{2}}du = \lim_{T\to 0^+}\int_{u=T}^{u=4} u^{-\frac{1}{2}}du =$

$\lim_{T\to 0^+} \frac{u^{\frac{1}{2}}}{\frac{1}{2}}\Big|_{u=T}^{u=4} = \lim_{T\to 0^+} 2\sqrt{u}\Big|_{u=T}^{u=4} = \lim_{T\to 0^+} 2\sqrt{4} - 2\sqrt{T} = 4$(D)

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(30) $\int_{\frac{1}{3}}^{\infty} \frac{3dx}{1+9x^2}$

Solution: Let $x = \frac{1}{3}\tan\theta$, so that $dx = \frac{1}{3}\sec^2\theta d\theta$. Then

$$\int_{x=\frac{1}{3}}^{x=\infty} \frac{3dx}{1+9x^2} = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \frac{3\frac{1}{3}\sec^2\theta d\theta}{1+\tan^2\theta^2} = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \frac{\sec^2\theta d\theta}{\sec^2\theta} = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} d\theta = \theta\Big|_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} = \frac{\pi}{4}\text{(F)}$$

The possible answers are:

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5 (F) $\frac{\pi}{4}$ (G) $\frac{\pi}{2}$ (H) $\frac{3\pi}{4}$ (I) $\frac{5\pi}{4}$ (J) $\frac{7\pi}{4}$

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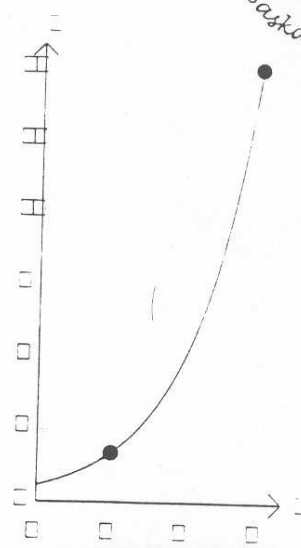
(31) A function satisfies the differential equation $y' = ky$. The graph of the function is also known to pass through the points (3,2) and (9,18). Find $y(0)$.



Solution: We must have $y(t) = y(0)e^{kt}$, and we know that

$y(3) = y(0)e^{k(3)} = 2$ and

$y(9) = y(0)e^{k(9)} = 18$.

Dividing the latter by the former, we get:
$\frac{y(9)}{y(3)} = \frac{y(0)e^{k(9)}}{y(0)e^{k(3)}} = \frac{e^{9k}}{e^{3k}} = e^{6k} = \frac{18}{2} = 9$, so $e^{6k} = 9$.

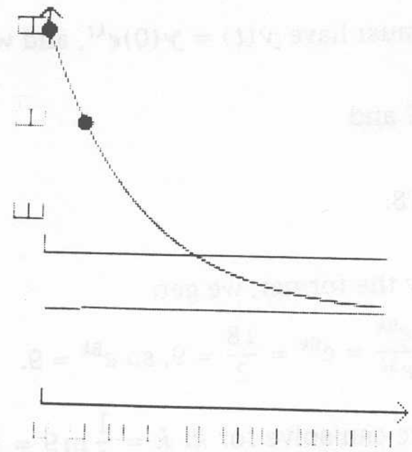
Taking logarithms, we can solve for k: $k = \frac{1}{6}\ln 9 = \frac{1}{3}\ln 3$. Substituting this back into the equation

$y(0)e^{k(3)} = 2$, we get $y(3) = y(0)e^{\frac{1}{3}\ln 3(3)} = 2$, or $y(3) = y(0)e^{\ln 3} = y(0)3 = 2$, so we have

$y(0) = \frac{2}{3}$(F)

The possible answers are:
(A) $\frac{1}{9}$ (B) $\frac{2}{9}$ (C) $\frac{1}{3}$ (D) $\frac{4}{9}$ (E) $\frac{5}{9}$ (F) $\frac{2}{3}$ (G) $\frac{8}{9}$ (H) 1 (I) $\frac{10}{9}$ (J) $\frac{11}{9}$

in a cooler whose temperature is 5°. At noon the temperature of the keg is observed to be 15°. How many hours AFTER 12 NOON will the beer be cold enough to serve(i.e., 8°)? **Hint:** The equation describing temperature change is $T(t) = T_\infty + (T_0 - T_\infty)e^{kt}$.



Solution: We have $T_\infty = 5$ and $T_0 = 20$, so the equation is $T(t) = 5 + 15e^{kt}$.

We also have $T(2) = 15$, so we can solve for k:

$T(2) = 15 = 5 + 15e^{k(2)}$ tells us that $e^{2k} = \frac{2}{3}$ and therefore $k = \frac{1}{2}\ln\frac{2}{3}$.

We substitute this into our equation:

$T(t) = 5 + 15e^{\frac{1}{2}\ln\frac{2}{3}t}$, and we want to know for what t we will have $T(t) = 8$, so we solve the equation

$8 = 5 + 15e^{\frac{1}{2}\ln\frac{2}{3}t}$ for t:

$\frac{1}{5} = e^{\frac{1}{2}\ln\frac{2}{3}t}$

$\ln 5 = \frac{1}{2}\ln\frac{2}{3}t$, $t = \frac{-\ln 5}{\frac{1}{2}\ln\frac{2}{3}} = 2\frac{\ln 5}{\ln\frac{3}{2}}$. Now this is the time it takes from 10A.M. to cool the beer, and we want to know how long after noon it will take, so we subtract 2 from this number:

$$2\frac{\ln 5}{\ln\frac{3}{2}} - 2 = 2\left(\frac{\ln 5}{\ln\frac{3}{2}} - 1\right) = 2\frac{\ln 5 - \ln\frac{3}{2}}{\ln\frac{3}{2}} = 2\frac{\ln\frac{5}{\frac{3}{2}}}{\ln\frac{3}{2}} = 2\frac{\ln\frac{2}{3}5}{\ln\frac{3}{2}} = \text{(D) } 2\frac{\ln\frac{10}{3}}{\ln\frac{3}{2}}$$

The possible answers are:

(A) $2\frac{\ln\frac{4}{3}}{\ln\frac{3}{2}}$ (B) $2\frac{\ln\frac{6}{3}}{\ln\frac{3}{2}}$ (C) $2\frac{\ln\frac{8}{3}}{\ln\frac{3}{2}}$ (D) $2\frac{\ln\frac{10}{3}}{\ln\frac{3}{2}}$ (E) $2\frac{\ln\frac{12}{3}}{\ln\frac{3}{2}}$ (F) $2\frac{\ln\frac{14}{3}}{\ln\frac{3}{2}}$ (G) $2\frac{\ln\frac{16}{3}}{\ln\frac{3}{2}}$